Prepotential approach to systems with dynamical symmetries

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Abstract

A prepotential approach to constructing the quantum systems with dynamical symmetry is proposed. As applications, we derive generalizations of the hydrogen atom and harmonic oscillator, which can be regarded as the systems with position-dependent mass. They have the symmetries which are similar to the corresponding ones, and can be solved by using the algebraic method.

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I. INTRODUCTION

It is known that the solvable models provide essential insights in the understandings of physical theory. Much effort has been made to find solvable models [1–7], and, in the meanwhile, various methods have also been proposed to solve systems, such as, the raising and the lowing operators method [8], factorization method [9], supersymmetric approach [10]. In this process, symmetries have been found to paly a fundamental role in the solvable systems, and knowledge of their presence in certain problems often simplifies the solutions considerably. All the well-known exactly solvable models as well as the quasi-exactly solvable models have specific symmetries in quantum mechanics. Among them, the hydrogen atom and the harmonic oscillator are the most important for their authenticity and simplicity. The dynamical symmetry of the N-dimensional hydrogen atom is described by the $SO(N + 1)$ Lie group [11, 12], and the N-dimensional harmonic oscillator is shown to have the $SU(N)$ dynamical symmetry [13, 14].

For our purpose, we would like to review some facts on the symmetries of the hydrogen atom and the harmonic oscillator. It is known that the classical orbits of these two systems are closed, and the conserved quantities responsible for the closeness of classical orbits have been proven to be the Rung-Lenz vector [15, 16] and the second order tensors [17] respectively. The symmetries are called dynamical symmetries because the nature of them is not geometrical but the symmetries in the phase space. These symmetries lead to an algebraic approach to determine the energy levels.

In this paper, we analyze the symmetries of the hydrogen atom and the harmonic oscillator, and try to propose some new quantum systems with dynamical symmetries. As the main contribution, we show that exactly solvable systems that follow specific dynamical symmetries can be constructed in a rather simple way. The idea is that we construct new generators of symmetric group from the known ones and get the corresponding Hamiltonian, which allow us to obtain exactly solvable systems. We call this constructing method as prepotential approach, which is motivated by the article [18]. The generators we supposed also contain undetermined functions. They are called the prepotential, a concept which plays a fundamental role in supersymmetric quantum mechanics [19, 20].

This paper is organized as follows. In Section 2, we give a description of the prepotential approach. In Section 3, we present an explicit construction of a Coulomb-like system. We

first construct the system in the 2-dimensional space. We then extend to the 3-dimensional space and in general to the N-dimensional space. In Section 4, we construct a 2-dimensional oscillator-like system and generalize it to the N-dimensional case. In the final section, we conclude with summary and some perspective.

II.   PREPOTENTIAL APPROACH

In this section, we will describe the idea that we use to construct some systems from the symmetry algebras directly. In fact, we illustrate this idea by considering the central potential in 2-dimensional space, and then an extension to the general N-dimension.

The central potential in 2-dimensional space have the conserved physical quantities: the angular momentum L_{12} (In general, the angular momentum is denoted $L_{ij} = x_i p_j - x_j p_i$; and L_{12} is the unique one in 2-dimensional space.) and the Hamiltonian H. Since H and L_{12} obey that

$$[L_{12}, H] = 0, \tag{1}$$

it follows that they have the common eigenstates. Let $|E, m\rangle$ be the eigenstates to eigenenergy E and the angular momentum eigenvalue m. We have

$$H|E, m\rangle = E|E, m\rangle, \quad L_{12}|E, m\rangle = m|E, m\rangle. \tag{2}$$

It is quite common that the Hamiltonian should be of the form

$$H = \frac{1}{4M(r)}p^2 + p^2\frac{1}{4M(r)} + h(L_{12}) + V(r),$$

where $M(r)$ is position-dependent effective mass and $h(L_{12})$ is an arbitrary function on L_{12}.

Motivated by the algebraic structure of the 2-dimensional hydrogen atom, we make the assumption that, besides the angular momentum L_{12}, there are some other physical quantities $T_\pm$ (In order to ensure the hermiticity, it is required that $T_+ = (T_-)^\dagger$.) that satisfy the following commutation relations

$$[L_{12}, T_\pm] = \pm T_\pm, \quad [T_+, T_-] = F(H, L_{12}) \tag{3}$$

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as well as the anticommutation relation

$$\{T_+, T_-\} = G(H, L_{12}),\tag{4}$$

where F and G are arbitrary functions on H and L_{12}. This enables us to construct systems with Lie algebra symmetries.

In view of (3) and (4), it is immediate that

$$T_+T_- = \frac{1}{2}(G(H, L_{12}) + F(G, L_{12})),\tag{5}$$
$$T_-T_+ = \frac{1}{2}(G(H, L_{12}) - F(G, L_{12})).$$

So,

$$\langle E, m|T_+T_-|E, m\rangle = \frac{1}{2}(G(E, m) + F(E, m)),\tag{6}$$
$$\langle E, m|T_-T_+|E, m\rangle = \frac{1}{2}(G(E, m) - F(E, m)).$$

Besides this, we know

$$T_+T_-T_+|E, m\rangle = T_+(T_-T_+)|E, m\rangle = (T_+T_-)T_+|E, m\rangle,\tag{7}$$
$$T_-T_+T_-|E, m\rangle = T_-(T_+T_-)|E, m\rangle = (T_-T_+)T_-|E, m\rangle,$$

In the above equations, we can see that the states $T_+|E, m\rangle$ and $T_-|E, m\rangle$ are the common eigenstates of L_{12} and H. Therefore, we obtain that

$$T_+|E, m\rangle = \sqrt{\frac{1}{2}\left(G(E, m) - F(E, m)\right)}|E', m+1\rangle,\tag{8}$$
$$T_-|E, m\rangle = \sqrt{\frac{1}{2}\left(G(E, m) + F(E, m)\right)}|E'', m-1\rangle,$$

The L_{12} span a finite-dimensional subspace. There exist a highest and a lowest weights for its representation, denoted by

$$T_+|E, \overline{m}\rangle = 0, \quad T_-|E, \underline{m}\rangle = 0.\tag{9}$$

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Therefore, from (8), we know

$$G(E, \overline{m}) - F(E, \overline{m}) = 0 \tag{10}$$

and

$$G(E, \underline{m}) + F(E, \underline{m}) = 0. \tag{11}$$

The degeneracy of states in this representation space should be a natural number, which leads to $\overline{m} - \underline{m} = n = 0, 1, 2 \ldots$.

The Hamiltonian can be obtained by comparing Eqs. (3) and (4) and adjusting the form of the $T_\pm$. The key ingredient of our approach is to choose suitable $T_\pm$. When $T_\pm$ are given, we can get the eigenenergy spectrum from Eqs. (10) and (11). It should be emphasized that $T_\pm$ are not required to be conserved quantities. While, we get the common eigenstates of L_{12} and H when they act on the state $T_+|E, m\rangle$ and $T_-|E, m\rangle$. The above idea will be demonstrated with concrete examples in Sections 3 and 4, where we will construct a Coulomb-like system and a oscillator-like system respectively.

III.   COULOMB-LIKE SYSTEM

The purpose of this section is to construct a system analogous to hydrogen atom by using the method described in the preceding section. We now give a brief description of the strategy that is used to construct the new system R_i $(i = 1, 2)$ from the original system.

In the hydrogen atom system, the two components of Runge-Lenz vector

$$R_1 = \frac{1}{2}(p_2 L_{12} + L_{12} p_2) - \frac{x_1}{r}, \qquad R_2 = -\frac{1}{2}(p_1 L_{12} + L_{12} p_1) - \frac{x_2}{r}$$

satisfy the commutation relations (in unit $\hbar = 1$)

$$[L_{12}, R_\pm] = \pm R_\pm, \quad [R_+, R_-] = -4HL_{12} \tag{12}$$

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and the anticommutation relation

$$\{R_+, R_-\} = (4L^2 + 1)H + 1, \tag{13}$$

where $R_\pm = R_1 \pm iR_2$, and the scalar L is defined as $L^2 = L_{ij}L_{ij}$.

We wish to construct new physical quantities that satisfy the following relations

$$[L_{12}, R_\pm] = \pm R_\pm, \quad [R_+, R_-] = F(H, L_{12}) \tag{14}$$

and the anticommutation relation

$$\{R_+, R_-\} = G(H, L_{12}). \tag{15}$$

To imitate the hydrogen atom, we assume that

$$R_1 = \frac{1}{2}(f(r)p_2L_{12} + L_{12}p_2f(r)) + g(r)x_1, \qquad R_2 = -\frac{1}{2}(f(r)p_1L_{12} + L_{12}p_1f(r)) + g(r)x_2 \tag{16}$$

which satisfy $[L_{12}, R_1] = iR_2, [L_{12}, R_2] = -iR_1$ (i.e. $[L_{12}, R_\pm] = \pm R_\pm$). The undetermined functions $f(r)$ and $g(r)$ are the prepotential. The anticommutation relation can be easily obtained

$$\begin{aligned}
\{R_+, R_-\} &= 2R_1^2 + 2R_2^2 \\
&= \frac{1}{2}\left(f(r)pf(r)p + pf^2(r)p + f(r)p^2f(r) + pf(r)pf(r) + 8f(r)g(r)\right)L_{12}^2 + 2g^2(r)r^2 \tag{17}
\end{aligned}$$

In order to get the relation (15), we see that $g^2(r)r^2$ must be a constant. For simplicity and consistency with the hydrogen atom, we select

$$g(r) = -\frac{1}{r}. \tag{18}$$

For brevity, we introduce a new operator π_i. Such an idea has also appeared in the paper [7]. Since π_i is the hermite operator, it can be written as

$$\pi_1 = \frac{1}{2}(f(r)p_1 + p_1f(r)), \quad \pi_2 = \frac{1}{2}(f(r)p_2 + p_2f(r)). \tag{19}$$

So R_1 and R_2 can be written as

$$R_1 = \frac{1}{2}(\pi_2 L_{12} + L_{12}\pi_2) - \frac{x_1}{r}, \quad R_2 = -\frac{1}{2}(\pi_1 L_{12} + L_{12}\pi_1) - \frac{x_2}{r}. \tag{20}$$

Equations (16) and (20) do not coincide. This is because (16) is the semiclassical attempt. In order to keep the symmetry and the hermiticity, we use the description (20).

Because $[\pi_1, \pi_2] = -i\frac{f(r)f'(r)}{r}L_{12}$, we have

$$[R_1, R_2] = -2iL_{12}\left(\frac{\pi^2}{2} - \frac{f(r)}{r} + \frac{f(r)f'(r)}{2r}L^2\right). \tag{21}$$

The expression in the bracket on the right-hand side of (21) is a function of H and L_{12}. So, we can take $f(r)f'(r) = \lambda r$, where λ is an arbitrary parameter. Then it follows that $f(r) = \sqrt{C + \lambda r^2}$, where C is an arbitrary constant. If we wish that the system reduces to the original hydrogen atom as $\lambda \to 0$, then we have

$$f(r) = \sqrt{1 + \lambda r^2}. \tag{22}$$

Now, we see that (21) has the following form

$$[R_1, R_2] = -2iL_{12}\left(\frac{\pi^2}{2} + \frac{\lambda L^2}{2} - \frac{\sqrt{1 + \lambda r^2}}{r}\right) \tag{23}$$

and the length of Rung-Lenz vector can be written as

$$R_1^2 + R_2^2 = 2L^2\left(\frac{\pi^2}{2} - \frac{\sqrt{1 + \lambda r^2}}{r}\right) + \frac{1}{2}\left(\frac{\pi^2}{2} - \frac{\sqrt{1 + \lambda r^2}}{r}\right) + \lambda L^2 + 1.$$

In free particle motion, we require conservation of linear momentum p_i is replaced by conservation of the vector π_i. This means the vector π_i is commute with the Hamiltonian when the potential vanishes. Therefore, the Hamiltonian of our constructed system is

$$H = \frac{\pi^2}{2} - \frac{\lambda L^2}{2} - \frac{\sqrt{1 + \lambda r^2}}{r}, \tag{24}$$

where $\frac{\pi^2}{2} - \frac{\lambda L^2}{2}$ is the term for kinetic energy, and the remaining is the potential. Note that this system can be considered as a hydrogen atom with a position-dependent effective mass

$M(r) = (1 + \lambda r^2)^{-1}$.

In view of (14) and (15), we get

$$F(H, L_{12}) = -4L_{12}(H + \lambda L^2) \tag{25}$$

and

$$G(H, L_{12}) = (2L^2 + \frac{1}{2})(2H + \lambda L^2) + 2\lambda L^2 + 2. \tag{26}$$

Substituting (25) and (26) into (10) and (11) and by a straightforward calculation, the eigenenergy spectrum of the system (24) is

$$E_n = -\frac{2}{(2n+1)^2} - \frac{1}{2}\lambda n(n+1) - 2\lambda n(n+1)\frac{2n+2}{(2n+1)^2}. \tag{27}$$

Now we extend the above to the 3-dimensional situation, which is more difficult than the 2-dimensional case. The new constructed physical quantity is a vector, which can be written as

$$\vec{R} = \frac{1}{2}(\vec{\pi} \times \vec{L} - \vec{L} \times \vec{\pi}) - \frac{\vec{r}}{r}, \tag{28}$$

where $\vec{L}$ is the vector $(L_1, L_2, L_3) = (L_{23}, L_{31}, L_{12})$ and $\vec{\pi}$ is the vector (π_1, π_2, π_3) with $\pi_i = \frac{1}{2}\left(\sqrt{1 + \lambda r^2}p_i + p_i\sqrt{1 + \lambda r^2}\right)$ $(i = 1, 2, 3)$.

Noting that

$$[L_\alpha, L_\beta] = i\epsilon_{\alpha\beta\gamma}L_\gamma, \quad [L_\alpha, \pi_\beta] = i\epsilon_{\alpha\beta\gamma}L\pi_\gamma, \quad [L_\alpha, R_\beta] = i\epsilon_{\alpha\beta\gamma}R_\gamma \quad \alpha, \beta, \gamma = 1, 2, 3$$

through a complicated computation, we obtain

$$\vec{R} \times \vec{R} = -2i(H + \lambda L^2)\vec{L}. \tag{29}$$

On the other hand, it is not hard to get the following

$$\vec{R} \cdot \vec{R} = (2H + \lambda L^2)L^2 + 2H + 2\lambda L^2 + 1. \tag{30}$$

The energy spectrum is similar to the 2-dimensional case, and it is written as

$$E_n = -\frac{1}{2(n+1)^2} - \frac{1}{2}\lambda n(n+2) - \lambda n(n+2)\frac{n+1}{(n+1)^2}. \tag{31}$$

In fact, we can also extend to the N-dimensional Coulomb-like system. It turns out that

$$
\begin{aligned}
[L_{ij}, R_k] &= i(\delta_{ik}R_j - \delta_{jk}R_i), \qquad [R_i, R_j] = -2i(H + \lambda L^2)L_{ij}, \\
R_i R_i &= (2H + \lambda L^2)L^2 + \frac{(N-1)^2}{2}(H + \frac{\lambda L^2}{2}) + \lambda L^2 + 1, \\
E_n &= -\frac{1}{2(n + \frac{N-1}{2})^2} - \frac{1}{2}\lambda n(n+N-1) - \lambda n(n+N-1)\frac{n+1}{(n + \frac{N-1}{2})^2},
\end{aligned}
\tag{32}
$$

where $R_i = \frac{1}{2}(L_{ij}\pi_j - \pi_j L_{ij}) - \frac{x_i}{r}$ is the conserved quantity.

Now we have completed the construction for the Coulomb-like system whose Hamiltonian is (24). Here, we would like to remark that it seems to be rather difficult to construct high-dimensional systems by directly using the prepotential approach because of the numerous amount of conserved quantities. However, as illustrated in this section, it is feasible to construct high-dimensional systems by first establishing the 2-dimensional case.

IV. OSCILLATOR-LIKE SYSTEM

In this section, we turn to the construction of oscillator-like system. To this end, we need to construct the physical quantities Q_{xy} and Q_1 which satisfy the following commutation relations

$$[Q_{xy}, L_{12}] = 2iQ_1, \qquad [Q_1, L_{12}] = -2iQ_{xy}, \qquad [Q_{xy}, Q_1] = F(H, L_{ij}), \tag{33}$$

and the anticommutation relation

$$Q_{xy}^2 + Q_1^2 = G(H, L_{ij}). \tag{34}$$

Analogous to the harmonic oscillator, we assume that

$$
\begin{aligned}
Q_{xy} &= \frac{1}{2}\left(f(r)p_1p_2 + p_1p_2 f(r)\right) + g(r)x_1 x_2, \\
Q_1 &= \frac{1}{2}g(r)(x_1^2 - x_2^2) + \frac{1}{4}\left(f(r)p_1^2 + p_1^2 f(r) - f(r)p_2^2 - p_2^2 f(r)\right).
\end{aligned}
\tag{35}
$$

Now we present the construction similar to the Coulomb-like system. By a direct computation, there holds that

$$
Q_{xy}^2 + Q_1^2 = \frac{1}{16}\left(f^2(r)p^2 + p^2 f^2(r)\right) + \frac{1}{4}g^2(r)r^2,
\tag{36}
$$

$$
[Q_{xy}, Q_1] = f(r)g(r)L_{12} + \frac{1}{4}f(r)G(x_1, p_1, x_2, p_2),
\tag{37}
$$

where $G(x_1, p_1, x_2, p_2)$ is a function of x_1, p_1, x_2, p_2, and is related to the function $f(r)$. We see that the right-hand side of equation (36) must be a function of H since L_{12} does not appear in it. In fact, because of the factor p^2, it is a reasonable assumption that it is just equal to H. Besides, we can see from (37) that $f(r)g(r)$ is a constant and $f(r)G(x_1, p_1, x_2, p_2)$ is a function of H and L_{12}.

Inspired by Coulomb-like system, we wish to obtain $Q_{x,y}$ and Q_1 of forms as given in (39). For this purpose, we choose $\pi_i = \frac{1}{2}\left(\sqrt{1 + \lambda r^2}\,p_i + p_i\sqrt{1 + \lambda r^2}\right)$ and thus

$$
f(r) = 1 + \lambda r^2 \quad \text{and} \quad g(r) = \frac{1}{1 + \lambda r^2}.
\tag{38}
$$

By the above analysis, we finish the constructions for the following quantities

$$
\begin{aligned}
Q_{xy} &= \frac{1}{2}\left(\pi_1\pi_2 + \pi_2\pi_1\right) + \frac{x_1 x_2}{1 + \lambda r^2}, \\
Q_1 &= \frac{1}{2}\frac{x_1^2 - x_2^2}{1 + \lambda r^2} + \frac{1}{2}(\pi_1^2 - \pi_2^2),
\end{aligned}
\tag{39}
$$

where the quantities $\vec{\pi}$ and L_{ij} are the same as those for hydrogen atom, namely,

$$
\begin{aligned}
\pi_i &= \frac{1}{2}(\sqrt{1 + \lambda r^2}\,p_i + p_i\sqrt{1 + \lambda r^2}), \\
L_{ij} &= x_i p_j - x_j p_i.
\end{aligned}
\tag{40}
\tag{41}
$$

These expressions are different from (35), but they have no effect on the conclusion.

By a simple calculation, it can be verified that

$$[Q_{xy}, L_{12}] = 2iQ_1, \qquad [Q_1, L_{12}] = -2iQ_{xy},$$

$$[Q_{xy}, Q_1] = -2iL_{12} + \frac{1}{2}i\lambda^2 L_{12} + i\lambda(2H + \lambda L^2)L_{12}, \tag{42}$$

and

$$Q_{xy}^2 + Q_1^2 = (H + \frac{\lambda L^2}{2})^2 + \lambda H + \frac{5}{4}\lambda^2 L^2 - L^2 - 1. \tag{43}$$

Similar to the system of Coulomb-like potential, the Hamiltonian of oscillator-like system is

$$H = \frac{\pi^2}{2} - \frac{\lambda L^2}{2} + \frac{\frac{1}{2}r^2}{1 + \lambda r^2}. \tag{44}$$

We next consider an extension to the N-dimensional case. Because the conserved quantities are not suitable to generalize to the N-dimensional space, we have to write them in another style. (In 2-dimensional case, we adopt the most common expression to facilitate the reader.) Besides the conservative angular momentum L_{ij}, we may define an extended quadrupole tensor

$$S_{ij} = \frac{1}{2}(\pi_i\pi_j + \pi_j\pi_i) + \frac{x_i x_j}{1 + \lambda r^2}. \tag{45}$$

The components have the commutators

$$[L_{ij}, S_{kl}] = 2i(-\delta_{jk}S_{il} - \delta_{jl}S_{ik} + \delta_{ik}S_{jl} + \delta_{il}S_{jk}) \tag{46}$$

and

$$[S_{ij}, S_{kl}] = i\Big[(1 - \frac{1}{4}\lambda^2)(L_{ik}\delta_{jl} + L_{jk}\delta_{jl} + L_{jk}\delta_{il} + L_{jl}\delta_{ik}) \tag{47}$$

$$-\frac{1}{2}\lambda(L_{ik}S_{jl} + L_{jk}S_{jl} + L_{jk}S_{il} + L_{jl}S_{ik}) - \frac{1}{2}\lambda(S_{jl}L_{ik} + S_{jl}L_{jk} + S_{il}L_{jk} + S_{ik}L_{jl})\Big].$$

The scalars formed from them are

$$I_1 = S_{ii} = 2H + \lambda L^2 \tag{48}$$

and

$$I_2 = S_{ij}S_{ji} - S_{ii}S_{jj} = -2L^2 - N(N-1) + \lambda \left[2(N-1)H - (N + \frac{1}{2})\lambda L^2 \right].$$

We can obtain the energy spectrum as given below

$$E_n = (n + \frac{N}{2})\sqrt{1 + \frac{1}{4}\lambda^2} - \frac{1}{2}\lambda(n^2 + Nn + \frac{N}{2}). \tag{49}$$

In fact, this system has been studied by using both the Lagrangian and the Hamiltonian formalisms in the paper [21]. While, compared with their method, our approach can give rise to the solution in a much simpler way. Besides, in our approach the integral is also not required.

V. SUMMARY

The prepotential approach to constructing models with dynamical symmetries is presented. We explicitly construct a Coulomb-like system and a oscillator-like system. We hope that our approach could give rise to new systems that cannot be dealt with by the known methods.

In the paper, we present this method in central field, and we can generalize it to the non-central field. We rewrite the two generators of all the three in central field. But in the non-central field, as the angular momentum is no longer a special conservation, all the generators should be rewritten. Moreover, we assume that the prepotential approach can be popularized to the Dirac equation. Because the Dirac Hamiltonian is said to have the spin or pseudospin symmetry corresponding to the same or opposite sign [22]. When the potentials are spherical, the total angular momentum can be divided into conserved orbital and spin parts, which form the $SU(2)$ algebra separately [23, 24]. We believe this approach can construct some other general classes of physical systems, especially the solvable models in quantum mechanics.

Acknowledgments

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